5.23

$$T[x+\epsilon h] - T[x] = \int_0^A dz \sqrt{\frac{1+x'^2+2x'h'\epsilon+\epsilon^2 h'^2}{z}} - \sqrt{\frac{1+x'^2}{z}}$$

$$= \int_0^A dz \sqrt{\frac{1+x'^2}{z}} \left(\sqrt{1 + \frac{2x'h'\epsilon+\epsilon^2 h'^2}{1+x'^2}} - 1 \right)$$

$$= \int_0^A dz \sqrt{\frac{1+x'^2}{z}} \left(\frac{2x'h'\epsilon+\epsilon^2 h'^2}{2(1+x'^2)} - \frac{4x'^2 h'^2 \epsilon^2}{8(1+x'^2)^2} + O(\epsilon^3) \right)$$

$$= \int_0^A dz \sqrt{\frac{1+x'^2}{z}} \left(\frac{(2x'h'\epsilon+\epsilon^2 h'^2)(1+x'^2) - x'^2 h'^2 \epsilon^2}{2(1+x'^2)^2} + O(\epsilon^3) \right)$$

and on a stationary path

$$= \frac{\epsilon^2}{2} \int_0^A dz \frac{h'^2}{\sqrt{z}(1+x'^2)^{3/2}} + O(\epsilon^3)$$

$$z = c^2 \sin^2\phi \qquad \frac{dz}{d\phi} = 2c^2 \sin\phi \cos\phi = c^2 \sin 2\phi$$

$$x = \tfrac{1}{2} c^2 (2\phi - \sin 2\phi) \qquad \frac{dx}{d\phi} = c^2 - c^2 \cos 2\phi$$

$$= \frac{\epsilon^2}{2} \int_0^{\phi_A} d\phi \, \frac{dz}{d\phi} \frac{h'(z)^2}{\sqrt{z}\left(1 + \left(\frac{dx}{dz}\right)^2\right)^{3/2}}$$

$$= \frac{\epsilon^2}{2}\int_0^{\phi_A} d\phi \;\; c^2 \sin 2\phi \;\; \frac{h'(z)^2}{c\sin\phi\left(1+\left(\frac{dx}{dz}\right)^2\right)^{3/2}}$$

$$= \epsilon^2 c \int_0^{\phi_A} d\phi \;\; \frac{\cos\phi \;\, h'(z)^2}{\left(1+\left(\frac{dx}{dz}\right)^2\right)^{3/2}}$$

$$\frac{dx}{dz} = \frac{1-\cos 2\phi}{\sin 2\phi}$$

$$\frac{1-2c+c^2+s^2}{s^2}$$

$$= \epsilon^2 c \int_0^{\phi_A} d\phi \; h'(z)^2 \frac{\cos\phi}{\left(\frac{2-2\cos 2\phi}{\sin^2 2\phi}\right)^{3/2}}$$

$$= \epsilon^2 c \int_0^{\phi_A} d\phi \;\; h'(z)^2 \frac{\cos\phi \, \sin^3 2\phi}{(2-2\cos 2\phi)^{3/2}}$$

$$= \epsilon^2 c \int_0^{\phi_A} d\phi \, h'(z)^2 \frac{\cos\phi \, . \, 8\cos^3\phi \, \sin^3\phi}{(4\sin^2\phi)^{3/2}}$$

$$= \epsilon^2 c \int_0^{\phi_A} d\phi \;\; h'(z)^2 \;\; \cos^4\phi \qquad > 0$$

$$T[x+\epsilon h] - T[x] > 0$$

$$T[x+\epsilon h] > T[x]$$

it's a global minimum.