Further Pure 1 Reducible Differential Equations

Use the substitution $x = e^u$, where u is a function of x, to find the general solution of the differential equation $x^2 \frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 6y = 0$

$$\frac{dx}{du} = e^u = x$$

$$\frac{dy}{du} = \frac{dx}{du}\frac{dy}{dx} = x\frac{dy}{dx}$$

$$\frac{d^2y}{du^2} = \frac{d}{du}\left(x\frac{dy}{dx}\right) = \frac{dx}{du}\frac{dy}{dx} + x\frac{d}{du}\left(\frac{dy}{dx}\right)$$

$$= x\frac{dy}{dx} + x\frac{dx}{du}\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - x\frac{dy}{dx}$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$$

$$x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 6y = 0 \Rightarrow \frac{d^2y}{du^2} - \frac{dy}{du} + 5\frac{dy}{du} + 6y = 0$$

$$\frac{d^2y}{du^2} + 4\frac{dy}{du} + 6y = 0$$

$$m^2 + 4m + 6 = 0 \Rightarrow m = \frac{-4 \pm \sqrt{16 - 24}}{2} = -2 \pm \sqrt{2}i$$

$$y = e^{-2u}\left(A\cos\sqrt{2}\,u + B\sin\sqrt{2}\,u\right)$$

$$y = x^{-2}\left(A\cos\left(\sqrt{2}\ln x\right) + B\sin\left(\sqrt{2}\ln x\right)\right)$$

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